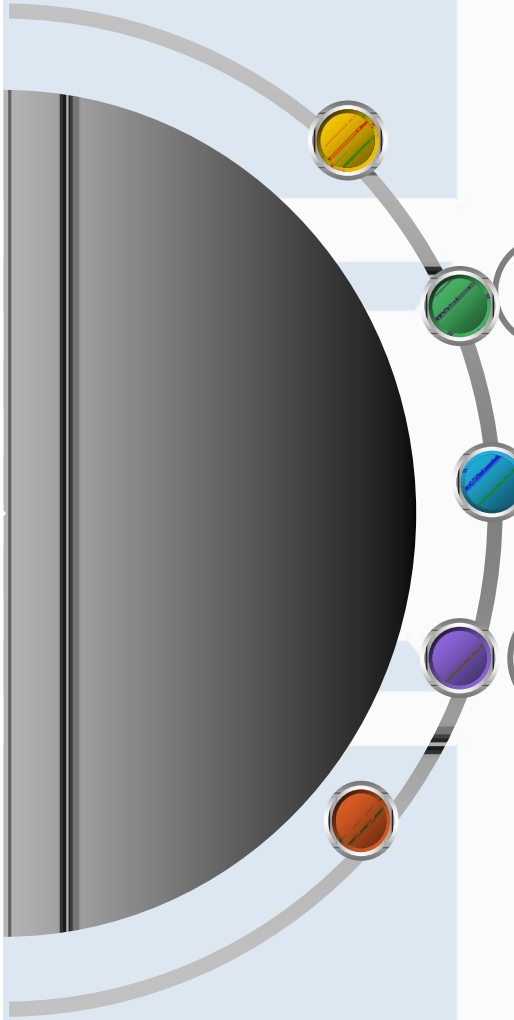
7.4 QR



7.4.1 Hessenberg

7.4.2 QR

7.4.3 QR

7.4.1 Hessenberg

7.9 Schur

$A \in R^{n\times n}$, Q

$$Q^T AQ = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ & R_{22} & \cdots & R_{2m} \\ & & \ddots & \vdots \\ & & & R_{mm} \end{pmatrix}$$

$R_{ii}(i = 1,2,\cdots,m)$

A A

A Schur

7.9

Schur

Hessenberg

7.2

$B = b_{ij} \in R^{n\times n}$ $b_{ij} > 0$

(i>j+1), B Hessenberg Hessenberg B

$$B = \begin{pmatrix} * & * & \cdots & \cdots & * \\ * & * & \cdots & \cdots & * \\ \ddots & \ddots & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & & * & * \end{pmatrix}$$

B $b_{k+1,k} = 0 \ 1 \le k \le n-1)$, B

Hessenberg

Hessenberg ,

7.3 $w \in R^n, \|w\|_2 = 1,$

$$H(w) = I - 2ww^T \qquad \textbf{7.4.1}$$

Householder

Houholder H=H w

(1) H $H = H^T = H^{-1}$ $H^T = H$

$w^T w = \|w\|_2 = 1$

$$H^T H = H^2 = I - 4ww^T + 4w(w^T w)w^T = I$$

(2) $x \in R^n, \quad y = Hx, \quad \|y\|_2 = \|x\|_2$

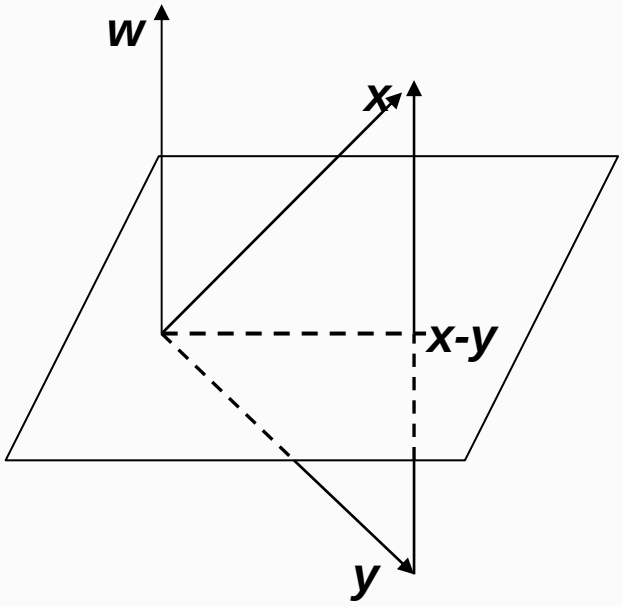
(3) S w , x y=Hx S ,

$$y = Hx = (I - 2ww^T)x$$

$$x - y = 2(w^T x)w$$

x-y w y x x y=Hx

x s 7-1



7-1

2

7.10 $x, y \in R^n, x \neq y \quad \|x\|_2 = \|y\|_2,$ H

$Hx=y$

$$w = (x-y)/\|x-y\|_2, H = I - 2ww^T \qquad \|w\|_2 = 1$$

$$x^T x = y^T y$$

$$2(x-y)^T x = x^T x - 2x^T y + y^T y = (x-y)^T (x-y) = \|x-y\|_2^2$$

$$Hx = x - 2ww^T x = x - \frac{2(x-y)(x-y)^T x}{\|x-y\|_2^2} = x - (x-y) = y$$

$$x = (x_1, x_2, \cdots, x_n)^T \neq 0 \qquad H$$

$$Hx = \sigma e_1 \qquad 7.4.2$$

$$\sigma = -sign(x_1)\|x\|_2, e_1 = (1,0,\cdots,0)^T \qquad H$$

$$\begin{cases} u = x - \sigma e_1, \\ \rho = \sigma(\sigma - x_1) \\ H = I - \rho^{-1} uu^T \end{cases} \qquad 7.4.3$$

σ ρ

7.4.2

7.4 $x=(3,5,1,1)^T$, H

$$Hx=-sign(x_1)\|x\|_2(1,0,0,0)^T$$

$$\|x\|_2=6,\sigma=-sign(x_1)\|x\|_2=-6,u=x-\sigma e_1=(9,5,1,1)^T$$

$\|u\|_2=108,\rho=54$ 7.4.3

$$H=I-\rho^{-1}uu^T=\frac{1}{54}\begin{pmatrix}-27&-45&-9&-9\\-45&29&-5&-5\\-9&-5&53&-1\\-9&-5&-1&53\end{pmatrix}$$

7.11 $A \in R^{n\times n}$, Q

$$B = Q^T AQ$$ **7.4.4**

Hessenberg

$A_1 = A$ a_1 A_1 $n-1$

7.4.2 $n-1$ H_1 $H_1 a_1 = \sigma_1 e_1,$ $e_1 =$

$(1,0,\cdots,0)^T \in R^{n-1}$ $P_1 = diag(1, H_1),$ P_1 $P_1^{-1} = P_1$

P_1 A_1

$$A_2 = P_1 A_1 P_1 = \begin{pmatrix} * & * & \cdots & * \\ \sigma_1 & * & \cdots & * \\ & * & \cdots & * \\ & \vdots & & \vdots \\ & * & \cdots & * \end{pmatrix}$$

a_2 A_2 2 $n-2$

$n-2$ H_2 $H_2 a_2 = \sigma_2 e_1,$ $e_1 = (1,0,\cdots,0,)^T \in R^{n-2}$ I_2

2 $P_2 = diag(I_2, H_2),$ P_2 P_2 A_2

$$A_3 = P_2 A_2 P_2 = \begin{pmatrix} * & * & * & \cdots & * \\ \sigma_1 & * & * & \cdots & * \\ & \sigma_2 & * & \cdots & * \\ & & * & \cdots & * \\ & & \vdots & & \vdots \\ & & * & \cdots & * \end{pmatrix}$$

n-2 Hessenberg

$$A_{n-1} = P_{n-2} A_{n-2} P_{n-2} = P_{n-2} \cdots P_2 P_1 A P_1 P_2 \cdots P_{n-2}$$

$B = A_{n-1}, Q = P_1 P_2 \cdots P_{n-2}$ 7.4.4

7.1 $A \in R^{n\times n}$, Q $B = Q^T AQ$

7.11 Hessenberg

Hessenberg

3 a_1 2

a_1 2

Q

7.4.2 QR

QR

Q

R

7.2 (QR) $A \in R^{n\times n}$

R, $A=QR$, R ,

7.11 A

A A QR

$$A = Q_1 R_1 = Q_2 R_2$$

R_1 R_2

$$Q_2^T Q_1 = R_2 R_1^{-1}$$

$$R_2 R_1^{-1\ T} = R_2 R_1^{-1\ -1}$$

$$D = R_2 R_1^{-1} = diag(d_1, d_2, \cdots, d_n),$$

$DD^T = D^2 = I,$ $d_i > 0, i = 1,2,\cdots,n$ $D = I$ $R_2 = R_1$

$Q_1 = Q_2$

A=QR R

$$R=(r_{ij}),$$

$$D=diag(\frac{r_{11}}{|r_{11}|},\frac{r_{22}}{|r_{22}|},\cdots\frac{r_{nn}}{|r_{nn}|}),$$

$\overline{Q}=QD$ $\overline{R}=D^{-1}R$ $|r_{ii}|$ $A=\overline{Q}\,\overline{R}$

7.12 QR

A QR $A=QR$ $B=RQ$ $B=Q^TAQ$ B

A B QR $A_1=A$

$$\begin{cases} A_k=Q_kR_k, \\ A_{k+1}=R_kQ_k, k=1,2,\cdots \end{cases}$$ **7.4.5**

7.4.5 $\{A_k\}$ QR

QR

7.13 **QR** $\{A_k\}$:

1 $A_{k+1} = Q_k^T A_k Q_k$;

2 $A_k = \bar{Q}_k \bar{R}_k$, $\bar{Q}_k = Q_1 Q_2 \cdots Q_k$ $\bar{R} = R_k \cdots R_2 R_1$

(1)

$$A_k = Q_{k-1}^T A_{k-1} Q_{k-1} = (Q_1 Q_2 \cdots Q_{k-1})^T A (Q_1 Q_2 \cdots Q_{k-1}) = Q_{k-1}^T A_k Q_{k-1},$$

$$\bar{Q}_k \bar{R}_k = Q_1 Q_2 \cdots Q_k R_k \cdots R_2 R_1 = \bar{Q}_{k-1} A_k \bar{R}_{k-1}$$

$$= \bar{Q}_{k-1} \bar{Q}_{k-1}^T A \bar{Q}_{k-1} \bar{R}_{k-1} = A \bar{Q}_{k-1} \bar{R}_{k-1}$$

$$\bar{Q}_1 \bar{R}_1 = Q_1 R_1 = A_1 = A \qquad 2$$

QR $\{A_k\}$

A

$\{A_k\}$

7.14 $A \in R^{n\times n}$,

$$|\lambda_1| > |\lambda_2| > \cdots |\lambda_n| > 0,$$

λ_i $x_i, i = 1,2,\cdots,n$ $X = (x_1, x_2, \cdots, x_n)$ X^{-1}

=LU L U QR $\{A_k\}$

$$\lim_{n\to\infty} a_{ii}^{(k)} = \lambda_i, i = 1,2,\cdots,n$$

QR $\{A_k\}$ Schur

Schur $\{A_k\}$ 2

7.5 QR

$$(1) = \begin{pmatrix} -3 & -5 & -1 \\ 13 & 13 & 1 \\ 13 & -5 & -1 \end{pmatrix}, (2) = \begin{pmatrix} 4 & 1 & -3 \\ -2 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

QR A Hessenberg A_1

$\{A_k\}$ 1

$$A_1 = \begin{pmatrix} -3.0000 & 4.3077 & -2.7282 \\ -13.9284 & 12.9738 & -5.5361 \\ & 0.4639 & 1.2062 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 10.1133 & 17.7711 & 0.8265 \\ -1.5511 & -0.6362 & -0.9381 \\ & 0.4656 & 1.5229 \end{pmatrix}$$

$$A_{16} = \begin{pmatrix} 6.0001 & 16.9820 & -9.2165 \\ 0.0000 & 3.0019 & -1.6317 \\ & 0.0012 & 1.9980 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 6.0000 & 16.9712 & -9.2364 \\ 0.0000 & 3.00001 & -1.6329 \\ & 0.0000 & 1.9999 \end{pmatrix}$$

A

2

$$A_1 = \begin{pmatrix} 4.0000 & -2.8284 & -1.4142 \\ 2.8284 & -1.0000 & -1.0000 \\ & -1.0000 & 1.0000 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2.3333 & -1.9379 & -5.1121 \\ 0.7454 & 1.2667 & 0.3266 \\ & -0.4899 & 0.4000 \end{pmatrix},$$

$$A_{25} = \begin{pmatrix} 2.0003 & -0.8171 & 3.6516 \\ 0.0002 & -0.3336 & 3.7263 \\ & -0.7456 & 2.3333 \end{pmatrix},$$

$$A_{26} = \begin{pmatrix} 2.0002 & -2.9999 & -2.2374 \\ 0.0001 & 2.9996 & 2.2366 \\ & -2.2349 & -0.9998 \end{pmatrix}$$

Schur 1 2

A_{25} A_{26} 2

$0.9999 \pm 1.0000\,i$

QR A Hessenberg H

H QR Hessenberg H

QR

i = 1,2,….n - 1, J i , i+1 H J i , i+1 H i

+1 i J(i , i+1) H i i+1

n-1 R $U^T H$ =R U^T =J

n-1,n J(n-2,n-1)…J(1,2) U^T Hessenberg U

Hessenberg H QR H=UR QR RU

RU Hessenberg QR H Hessenberg

7.6 Hessenberg

$$H = \begin{pmatrix} 5 & -2 & -5 & -1 \\ 1 & 0 & -3 & 2 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$H_1 = H$ H_1 J 12 J 23 J 34

$$H_1 = U_1 R_1 = \begin{pmatrix} 0.9806 & -0.0377 & 0.1923 & -0.1038 \\ 0.1961 & 0.1887 & -0.8804 & -0.4192 \\ 0 & 0.9813 & 0.1761 & 0.0740 \\ 0 & 0 & 0.3962 & -0.8989 \end{pmatrix} \bullet \begin{pmatrix} 5.0992 & -1.9612 & -5.4912 & -0.3922 \\ 0 & 2.0381 & 1.5852 & -2.5288 \\ 0 & 0 & 2.5242 & -3.2736 \\ 0 & 0 & 0 & 0.7822 \end{pmatrix}$$

$U_1^T = J$ 34 J 23 J 12 U_1 R_1 H_2

$$H_2 = R_1 U_1 = \begin{pmatrix} 4.6157 & 5.9508 & 1.5922 & 0.2390 \\ 0.3997 & 1.9401 & -2.5171 & 1.5361 \\ 0 & 2.4770 & -0.8525 & 3.1294 \\ 0 & 0 & 0.3099 & -0.7031 \end{pmatrix} \circ$$

11

$$H_{12} = \begin{pmatrix} 4.0000 & * & * & * \\ & 1.8789 & -3.5910 & * \\ & 1.3290 & 0.1211 & * \\ & & & -1.0000 \end{pmatrix} \circ$$

4 -1

$$\begin{vmatrix} 1.8789-\lambda & -3.5910 \\ 1.3290 & 0.1211-\lambda \end{vmatrix} = 0$$

$1 \pm 2i$ H

$$\lambda^4 - 5\lambda^3 + 7\lambda^2 - 7\lambda - 20 = 0,$$

QR

7.4.3 QR

QR

Hessenberg QR QR

QR

$$\begin{cases} A_1 \quad A \quad Hessenberg \\ A_k - s_k I = Q_k R_k \quad QR \\ A_{k+1} = R_k Q_k + s_k I \end{cases} \qquad k = 1,2,\cdots$$

$, A_k$ A_{k+1} QR 。

$$R_kQ_k + s_kI = Q_k^TQ_kR_kQ_k + s_kQ_k^TQ_k = Q_k^T(Q_kR_k + s_kI)Q_k, \qquad A_{k+1} = Q_k^TA_kQ_k。$$

A_k A_1 A

$s_1, s_2, \cdots, s_k \cdots,$ *Hessenberg*

$\{A_k\}$ $A_k = \left(a_{ij}^{(k)}\right)_{n\times n}, B_k$ A_k $n-1$

$s_k = a_{nn}^{(k)},$ A_k $a_{nn}^{(k)}$ A

$B_k \in R^{(n-1)\times(n-1)}$ QR

A

$a_{n,n-1}^{(k)}$ $\left|a_{n,n-1}^{(k)}\right| \le \varepsilon \|A_1\|_\infty$

$\left|a_{n,n-1}^{(k)}\right| \le \varepsilon\left(\left|a_{n-1,n-1}^{(k)}\right| + \left|a_{nn}^{(k)}\right|\right),$

$a_{n,n-1}^{(k)}$

$a_{n,n-1}^{(k)} = 0, a_{nn}^{(k)}$

ε

A

QR

(1) $s_k = a_{nn}^{(k)}$

(2) s_k $\begin{pmatrix} a_{n-1,n-1}^{(k)} & a_{n-1,n}^{(k)} \\ a_{n,n-1}^{(k)} & a_{nn}^{(k)} \end{pmatrix}$

$a_{nn}^{(k)}$

Hessenberg A_1 *QR*

$$P_{n-1,n} \cdots P_{23} P_{12} (A_1 - s_1 I) = R_1,$$

$$A_2 = R_1 P_{12}^T P_{23}^T \cdots P_{n-1,n}^T + s_1 I$$

A_2 *Hessenberg*

7.7

QR

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 1 \\ 1 & 1 & -6 \end{pmatrix}。$$

A Hessenberg 7.4.3

$$H = I - 2ww^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2/\sqrt{5} & -1/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2\sqrt{5} \end{pmatrix},$$

$$A_1 = H^T AH = \begin{pmatrix} -1 & -\sqrt{5} & 0 \\ -\sqrt{5} & -3.6 & 0.2 \\ 0 & 0.2 & -6.4 \end{pmatrix}。$$

$s_1 = -6.5,$

$\theta_1 = -0.392590761, \theta_2 = 0.114997409,$

$$P_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix},$$

$$R_1 = P_2 P_1 (A_1 + 6.4I) = \begin{pmatrix} 5.844655679 & -3.137183510 & -0.076516671 \\ & 1.743000879 & 0.183563711 \\ & & -0.021202899 \end{pmatrix},$$

$$A_2 = R_1 P_1^T P_2^T - 6.4I = \begin{pmatrix} 0.200234192 & -0.666846149 & 0 \\ -0.666846149 & -4.779171336 & -0.002432908 \\ 0 & -0.002432908 & -6.421062856 \end{pmatrix}。$$

$$A_3 = \begin{pmatrix} 0.283205888 & -0.157002612 & 0 \\ -0.157002612 & -4.862139274 & 0.000000006 \\ 0 & 0.000000006 & -6.421066615 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0.287735078 & -0.036401350 & 0 \\ -0.036401350 & -4.866668465 & 0 \\ 0 & 0 & -6.421066615 \end{pmatrix}。$$

$\lambda_3 = -6.421066615$ 2×2 $\lambda_1 = 0.287992138$

$\lambda_2 = -4.866925525$

2×2 $s_1 = -6.469693846$

$$A_2 = \begin{pmatrix} 0.194154158 & -0.689146437 & 0 \\ -0.689146437 & -4.773105873 & 0.005374767 \\ 0 & 0.005374767 & -6.421048287 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.282852106 & -0.162696110 & 0 \\ -0.162696110 & -4.861785493 & 0.000011074 \\ 0 & 0.000011074 & -6.421066615 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0.287716058 & -0.037723983 & 0 \\ -0.037723983 & -4.866649445 & 0 \\ 0 & 0 & -6.421066615 \end{pmatrix}$$

$\lambda_1 = 0.287992139$ $\lambda_2 = -4.866925525$ $\lambda_3 = -6.421066615$

QR QR

$$A_2 = \begin{pmatrix} -4.838383838 & 0.552770798 & 0 \\ 0.552770798 & -0.439393939 & -2.004127972 \\ 0 & -2.004127972 & -5.727272727 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} -5.282161439 & 0.687687671 & 0 \\ 0.687687671 & -6.005830703 & -0.000000190 \\ 0 & -0.000000190 & 0.287992139 \end{pmatrix},$$

$$A_{30} = \begin{pmatrix} -6.421054217 & 0.004390120 & 0 \\ 0.004390120 & -4.866937928 & 0 \\ 0 & 0 & 0.287992139 \end{pmatrix}$$

A_7

$\lambda_3 \approx 0.287992139$ λ_1 λ_2 A_7 2×2

$\lambda_1 \approx -6.421066617$ $\lambda_2 \approx 4.866925525$